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Space-Times with Covariant-Constant Energy-Momentum Tensor

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It is shown that a general relativistic space-time with covariant-constant energy-momentum tensor is Ricci symmetric. Two particular types of such general relativistic space-times are considered and the nature of each is determined.

1. INTRODUCTION

General relativity flows from the Einstein equation which implies that the energy-momentum tensor is of vanishing divergence. This requirement of the energy-momentum tensor is satisfied if this tensor is covariant-constant. It is therefore meaningful to ask whether the energy-momentum tensor of a given general relativistic space-time is covariant-constant. In this paper we first show that a general relativistic space-time with covariant-constant energy-momentum tensor is Ricci symmetric, i.e., it has covariant-constant Ricci tensor. Next we consider a special type of space-time which is called pseudo Ricci symmetric.

A Riemannian manifold (M^n, g) is called pseudo Ricci symmetric if its Ricci tensor S of type (0, 2) satisfies the condition (Chaki, 1988)

$$(\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X) \tag{1.1}$$

where A is a 1-form,

$$g(X, P) = A(X) \tag{1.2}$$

for all vector fields X, and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

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A is called the associated 1-form and P is called the basic vector field of such a manifold, and an n-dimensional manifold of this kind is denoted by $(PRS)_n$.

It is shown that if a general relativistic space-time is a semi-Riemannian $(PRS)_4$ with covariant-constant energy-momentum tensor, then the space-time is a vacuum, i.e., devoid of matter.

Finally, a general relativistic perfect fluid space-time with cosmological constant λ and flow vector field U is considered in which the condition $B(R(X, Y, Z)) = 0$ is satisfied, where $B(X) = g(X, U)$ for all vector fields X. It is shown that if in such a space-time the energy-momentum tensor is covariant-constant, then each of $\nabla_U U$ and div U is zero and λ satisfies the condition $r/6 < \lambda < r/2$. In other words, the acceleration vector and the expansion scalar of the fluid are zero and the cosmological constant λ satisfies the condition $r/6 < \lambda < r/2$.

2. GENERAL RELATIVISTIC SPACE-TIME WITH COVARIANT-CONSTANT ENERGY-MOMENTUM TENSOR

Let (M^4, g) be a general relativistic space-time and T denote the (0, 2) type of energy-momentum tensor. In this section we suppose that

$$\nabla T = 0 \tag{2.1}$$

where ∇ has the meaning already mentioned. Denote the scalar curvature of (M^4, g) by r. Then Einstein's equation can be written as

$$S - \frac{1}{2} rg = kT \tag{2.2}$$

where k is the gravitational constant. Differentiating (2.2) covariantly, we get

$$\nabla S - \left(\frac{1}{2} dr\right)g = k\nabla T \tag{2.3}$$

In virtue of (2.1) it follows from (2.3) that

$$\nabla S - \left(\frac{1}{2} dr\right)g = 0 \tag{2.4}$$

Contracting (2.4), we have

$$dr - 2dr = 0$$

or,

$$dr = 0 \tag{2.5}$$

In virtue of (2.5), equation (2.4) takes the form

$$\nabla S = 0 \tag{2.6}$$

This shows that the space-time under consideration has covariant-constant Ricci tensor, i.e., the space-time is Ricci symmetric.

Hence we can state the following result.

Theorem 1. A general relativistic space-time with covariant-constant energy-momentum tensor is Ricci symmetric and is of constant scalar curvature.

Note. Theorem 1 of Garcia de Andrade (1991) follows as a particular case of the above theorem.

3. PSEUDO-RICCI-SYMMETRIC GENERAL RELATIVISTIC SPACE-TIME WITH COVARIANT-CONSTANT ENERGY-MOMENTUM TENSOR

In this section we consider a general relativistic space-time which is of $(PRS)_4$ type with associated 1-form A and basic vector field P. We further suppose that $\nabla T = 0$. Then from Theorem 1 we get

$$\nabla S = 0 \tag{3.1}$$

Since the space-time is of type $(PRS)_4$, we obtain

$$(\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X) \tag{3.2}$$

In virtue of (3.1), the relation (3.2) takes the form

$$2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X) = 0 \tag{3.3}$$

It is known (Chaki, 1988) that in a $(PRS)_n$ with basic vector field P the relation $S(X, P) = 0$ holds for all vector fields X. This result holds also for a semi-Riemannian $(PRS)_n$. Therefore for the space-time $(PRS)_4$ under consideration we have

$$S(X, P) = 0 \tag{3.4}$$

for all vector fields X. Putting $Z = P$ in (3.3) and taking account of (3.4), we get

$$A(P)S(Y, X) = 0 \tag{3.5}$$

From (3.5) it follows that

$$S(Y, X) = 0 \tag{3.6}$$

From (3.6) we have

$$r = 0 \tag{3.7}$$

In virtue of (3.6) and (3.7) it follows from Einstein's equation (2.2) that

$$T = 0 \tag{3.8}$$

This means that the space-time is devoid of matter. This leads to the following result.

Theorem 2. A pseudo-Ricci-symmetric relativistic space-time with covariant-constant energy-momentum tensor is a vacuum.

4. PERFECT FLUID SPACE-TIME WITH COSMOLOGICAL CONSTANT IN WHICH THE ENERGY-MOMENTUM TENSOR IS COVARIANT-CONSTANT AND $B(R(X, Y, Z)) = 0$, WHERE $B(X) = g(X, U)$ FOR ALL VECTOR FIELDS X, WITH U THE FLOW VECTOR FIELD OF THE FLUID

Denote the cosmological constant by λ; then Einstein's equation can be written as follows (Beem and Ehrlich, 1981):

$$S - \frac{1}{2} rg + \lambda g = kT \tag{4.1}$$

where

$$T = (\sigma + p)B \otimes B + pg \tag{4.2}$$

with σ and p denoting the density and pressure of the fluid, respectively, and B being given by

$$g(X, U) = B(X) \qquad \text{for all} \quad X \tag{4.3}$$

We can express (4.1) as follows:

$$S(X, Y) - \frac{1}{2} rg(X, Y) + \lambda g(X, Y) = k[(\sigma + p)B(X)B(Y) + pg(X, Y)] \tag{4.4}$$

By hypothesis,

$$B(R(X, Y, Z)) = 0$$

or

$$'R(X, Y, Z, U) = 0 \tag{4.5}$$

where

$$'R(X, Y, Z, U) = g[R(X, Y, Z), U] \tag{4.6}$$

Taking a frame field and contracting (4.5), we get

$$S(X, U) = 0 \tag{4.7}$$

Now, putting $Y = U$ in (4.4), we get

$$\begin{aligned} S(X, U) - \frac{1}{2} rg(X, U) + \lambda g(X, U) \\ = k[(\sigma + p)B(U)B(X) + pg(X, U)] \end{aligned} \tag{4.8}$$

In virtue of (4.7) and taking account of the fact that $B(U) = -1$ because U is timelike, we can write (4.8) as follows:

$$-\frac{1}{2} r + \lambda = k[-(\sigma + p) + p] = -k\sigma$$

Hence

$$\sigma = \frac{r - 2\lambda}{2k} \tag{4.9}$$

Again taking a frame field and contracting (4.4), we get

$$r - 2r + 4\lambda = k(-\sigma - p + 4p) = k(3p - \sigma)$$

Hence

$$\begin{aligned} 3kp &= k\sigma - r + 4\lambda = \frac{r - 2\lambda}{2} - r + 4\lambda \\ &= 3\lambda - \frac{r}{2} = \frac{6\lambda - r}{2} \end{aligned}$$

From this we get

$$p = \frac{6\lambda - r}{6k} \tag{4.10}$$

By hypothesis, $\nabla T = 0$. Hence from Theorem 1 it follows that r is constant. Therefore from (4.9) and (4.10) we see that both σ and p are constant.

It is known (O'Neill, 1983) that the equation div $T = 0$ implies the following for a perfect fluid:

$$U\sigma = -(\sigma + p) \operatorname{div} U \qquad \text{(energy equation)} \tag{4.11}$$

$$(\sigma + p)\nabla_U U = -\operatorname{grad} p - (Up)U \qquad \text{(force equation)} \tag{4.12}$$

Since in this case both σ and p are constant, it follows from (4.11) and (4.12) that

$$\operatorname{div} U = 0 \qquad \text{and} \qquad \nabla_U U = 0$$

But div U represents the expansion scalar and $\nabla_U U$ represents the acceleration vector.

Thus in this case both the expansion scalar and the acceleration vector are zero.

Summing up, we can state the following result:

Theorem 3. Let a perfect fluid space-time with cosmological constant λ and flow vector field U satisfy the condition $B(R(X, Y, Z)) = 0$, where $g(X, U) = B(X)$ for all X. If in such a space-time the energy-momentum tensor is covariant-constant, then the fluid has vanishing acceleration and its expansion scalar is zero. Further, in this case the cosmological constant has to satisfy the condition $r/6 < \lambda < r/2$.

REFERENCES

Beem, J. K., and Ehrlich, P. L. (1981). *Global Lorentzian Geometry*, Dekker, New York.

Chaki, M. C. (1988). On pseudo Ricci symmetric manifolds, *Bulgarian Journal of Physics*, **15**, 526.

Garcia de Andrade, L. C. (1991). *International Journal of Theoretical Physics*, **30**, 235.

O'Neill, B. (1983). *Semi-Riemannian Geometry*, Academic Press, New York.